# Outline of Solutions to Exam in Financial Econometrics A: January 2013 (for fall term 2012)

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## Question A:

**Question A.1: Solution:** $\gamma = \omega / (1-\alpha)$ and hence$\omega = \gamma (1-\alpha)$.

**Question A.2: Solution:** Drift function with $\delta(x) = 1+x^4$. $E(\delta(x_t) | x_{t-1} = x) = 3\left(\gamma(1-\alpha) + \alpha x^2\right)^2$ and standard arguments give the desired as the dominating term is $3\alpha^2$ for which we need $\alpha$ such that we can find a $\beta$ that satisfies: $3\alpha^2 < \beta < 1$.

**Question A.3:Solution:** $l_t(\theta) = \log \sigma_t^2 + \frac{x_t^2}{\sigma_t^2}$ and direct differentiation gives the desired.

**Question A.4: Solution:** $E\left( \left. \frac{\partial l_t(\theta)}{\partial \alpha} \right|_{\theta=\theta_0} \middle| x_{t-1} \right) = E\left(1 - z_t^2\right) \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\sigma_{0t}^2} = 0$ and $\frac{\partial l_t(\theta)}{\partial \alpha}$ is a MGD. Hence as

$$\frac{1}{T} \sum_{t=1}^{T} E\left( \left. \frac{\partial l_t(\theta)}{\partial \alpha} \right|_{\theta=\theta_0} \middle| x_{t-1} \right) = 2 \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\sigma_{0t}^2} \right)^2 \to^P \xi := 2E\left( \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\sigma_{0t}^2} \right)^2 < \infty.$$

This is finite either since $Ex_t^4 < \infty$ (strong requirement) or inequality (zero requirement):

$$\left( \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\sigma_t^2} \right)^2 = \left( \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\gamma_0 + \alpha_0 \left(x_{t-1}^2 - \gamma_0\right)} \right)^2 \leq \frac{1}{\gamma_0^2}.$$

**Question A.5: Solution:** LLN: $x_t$ is weakly mixing, and provided $\alpha_0 < 1$ gives: $\hat{\gamma}_T \to^P Ex_t^2 = \gamma_0$.

One may add (not asked for) that using MGD CLT gives provided $4^{th}$ order moments finite,

$$\sqrt{T}\left(\hat{\gamma}_{VT}-\gamma_0\right)=\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_t^2-\gamma_0\right)\rightarrow^{D}N\left(0,\psi\right).$$

**Question A.6:Solution:**

Normality strongly rejected - while OK in terms of ARCH effects. $\hat{\gamma}_{VT}$ is a naive and model free variance estimator - and hence would be consistent provided series second order moment. However, as model is misspecified we do not know if $\hat{\alpha}=0.6$ is "small or big". Would be ok if $z_t$ Gaussian - Another model: Probably allow for fatter tails - $t_v$ distribution instead of Gaussian. Can also then work out conditions for finite moments (variance).

# Question B:

**Question B.1: Solution:** That while the residuals are well-specified (no ARCH, and normality accepted) - the states are very close to be absorbing and hence a degenerate MS model.

**Question B.2: Solution:** This is a mixture ARCH - most likely more fat tails than classic ARCH, and classic iid mixture. Likelihood, $L_T\left(\theta\right)=\sum_{t=1}^{T}l_t\left(\theta\right),$

$$l_t\left(\theta\right)=\log\left(pf_1\left(y_t|y_{t-1}\right)+\left(1-p\right)f_2\left(y_t|y_{t-1}\right)\right),$$

where $f_1$ and $f_2$ are Gaussian densities for the normal distribution with mean zero, and variances $\sigma_{1t}^2$ and $\sigma_{2t}^2$ respectively. For example, up to a constant,

$$\log f_1=-\tfrac{1}{2}[\log\sigma_{1t}^2+y_t^2/\sigma_{1t}^2]$$

**Question B.3: Solution:** Same as EM-algorithm: One should here explain main principles of EM-algorithm.

Alternative: Discuss how MLE using numerical optimization will work.

Some computations/arguments may be of the kind:

Fix $p$ and $\alpha$ at some initial value, $p_{(1)}$ and $\alpha_{(1)}$ say. Next compute $p_t^*\left(p_{(1)},\alpha_{(1)}\right)$ find the $\alpha_{(2)}$ which solves ("weighted ARCH MLE")

$$\partial L_T/\partial\alpha=\sum_{t=1}^{T}p_t^*\left(p_{(1)}\right)\left(1-\tfrac{y_t^2}{\sigma_{1t}^2}\right)\tfrac{y_{t-1}^2}{\sigma_{1t}^2}=0,$$

and the $p_{(2)}$ which solves $\partial L_T/\partial p=0$ (nonlinear). Next, update $p_t^*=p_t^*\left(p_{(2)},\alpha_{(2)}\right)$ and repeat finding zero scores.

$$\begin{aligned}\partial l_t/\partial p &= \tfrac{f_1-f_2}{pf_1+(1-p)f_2}\rightarrow\partial L_T/\partial p=0=\sum\left(\tfrac{f_{1t}-f_{2t}}{pf_{1t}+(1-p)f_{2t}}\right)\ \text{(non-linear)}\\ \partial l_t/\partial\alpha &= \left(\tfrac{p_1f_1}{pf_1+(1-p)f_2}\right)\left(1-\tfrac{y_t^2}{\sigma_{1t}^2}\right)\tfrac{y_{t-1}^2}{\sigma_{1t}^2}\ \text{("weighted"-ARCH)}\end{aligned}$$

**Question B.4: Solution:** If $p = 1$ the model reduces to an ARCH(1) - so with $\hat{\omega}$ and $\hat{\alpha}$ the MLE's in this case would use:

$$y_t / \sqrt{\hat{\omega} + \hat{\alpha} y_{t-1}} \simeq z_t \overset{D}{=} N(0,1).$$

And the usual formula from lectures can be written here $(-\sqrt{\hat{\omega} + \hat{\alpha} y_{t-1}} \Phi^{-1}_{5\%})$.

Comparing the iid with $\mathrm{cVaR}_t$ : Discuss mixed Gaussian (fatter tails marginally).

**Question B.5: Solution:** In this case one can use the smoothed $s_t$ values, $E(s_t = 1 | y_1, .., y_t) = p_t^*$, which is part of the "usual" output from MS estimation and would also be here by definition. Hence with $\hat{p}_t^*$ one could use (other ideas may also be proposed such as conditioning on $y_1, ..., y_T$ instead):

$$y_t / \sqrt{\hat{p}_t^* (\hat{\omega} + \hat{\alpha} y_{t-1}) + (1 - \hat{p}_t^*) \hat{\gamma}} \simeq z_t \overset{D}{=} N(0,1).$$